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This issue of
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is
dedicated in honour
of the
department of statistics.

CONTENTS

Editor's Note

This issue of the Journal contains five articles. The issue contains spectrum of papers covering the topics Fuzzy Number and Partial Fuzzy Number, A Few Words on $M/E_k/1$ Queuing Model with Balking and Reneging Customer, An article on Actuarial science, Paper on The Population Model Using Fuzzy data, Comparison of Normal CDF approximations.

This issue of the journal have been brought out for the first time and this is the 1st issue. We are thankful to all the members of the teaching staff of Statistics Department for their help in bringing out the issue. We are grateful to Pragjyotish College for allowing us to publish the journal. We acknowledge the help received by Dr. Dayananda Pathak, Principal of Pragjyotish College.

We hope mosaic of articles published in this journal will help the younger generation students to grow an interest and knowledge of the subject.

Editor

Dr. Pranita Goswami

OCTOBER, 05, 2010

Fuzzy Number and Partial Fuzzy Number

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Abstract: The Fuzzy Number can be Partial Fuzzy Number if we take a portion of Fuzzy number which is again a Fuzzy number. The base line is shifted from 0 to 1 to any of its α cuts .In otherwords instead of taking the whole Fuzzy number we take a portion of it. In this paper we have fuzzified a portion of the triangular number by interval approach

Key Words: Partial fuzzy number, triangular number,baseline

1. **Introduction**: Fuzzy number is an interval.Its membership function is from 0 to1or in terms of α cut..For eg. 2.0 is a number but it will be fuzzy if is taken over an interval [1.9,2.0, 2.1]. Its value is 1 in 2.0 and zero at 1.9 and 2.1.In other words it is exact at 2.0 and fuzzy over the interval [1.9,2.0] and {2.0,2.1].Fuzzy number can be linear or nonlinear. Partial fuzzy number is a fuzzy number which is a portion of the original fuzzy number which is obtained by changing the base line.In other words the base line is at the ά cut. The partial fuzzy number is important as the whole triangular fuzzy number may not be required but only a portion of it is required.Workson fuzzy statistics have started within the last few years (see eg .Klior and Yuan(1997), Goswami and Baruah (2008) Kaufmann and Gupta(1985)).We have found that no work has yet been done on partial fuzzy number by using fuzzy approach. We put forward one such example in this paper.

2.**Triangular Fuzzy Number**:The triangular fuzzy number in terms of membership function for A is represented by

$$\mu_A(x)=\begin{cases}0, & x<-1\\ \dfrac{x+1}{2} & ,-1\le x\le 1\\ \dfrac{3-x}{2} & ,1\le x\le 3\\ 0, & x>3\end{cases} \qquad \ldots(1)$$

And the triangular fuzzy number in terms of membership function for B is represented by

$$\mu_B(x)=\begin{cases}0, & x<1\\ \dfrac{x-1}{2} & ,1\le x\le 3\\ \dfrac{5-x}{2} & ,3\le x\le 5\\ 0, & x>5\end{cases} \qquad \ldots(2)$$

From (1)

$$\alpha=\frac{a_1^{(\alpha)}}{2}+\frac{1}{2} \qquad \alpha=-\frac{a_2^{(\alpha)}}{2}+\frac{3}{2}$$

Where the confidence interval for A is

$$A_{(\alpha)}=\left[a_1^{(\alpha)},a_2^{(\alpha)}\right]$$

$$=[2\alpha-1,-2\alpha+3]$$

Using(2)

$$\alpha=\frac{b_1^{(\alpha)}-1}{2} \qquad \alpha=-\frac{b_2^{(\alpha)}}{2}+\frac{5}{2}$$

Hence

$$B_{(\alpha)}=\left[B_1^{(\alpha)},B_2^{(\alpha)}\right]$$

$$= [2\alpha+1, -2\alpha+5]$$

We obtain the interval for $A_\alpha + B_{(\alpha)}$ as

$$A_{(\alpha)} + B_{(\alpha)} = [2\alpha-1, -2\alpha+3](+)[2\alpha+1, -2\alpha+5]$$

$$= [4\alpha, -4\alpha+8]$$

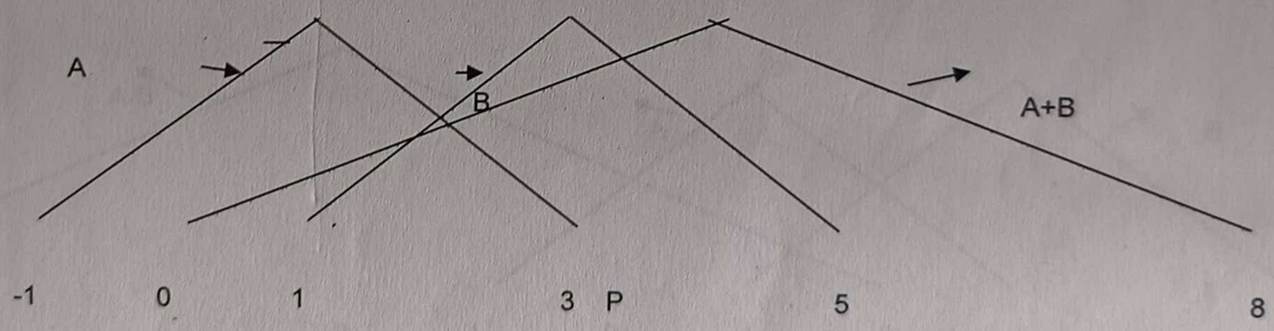
We now have two equations to solve viz

$X_1 = 4\alpha$ and $X_2 = -4\alpha+8$

Thus the addition of two fuzzy number in terms of membership function is represented by

$$\mu_{A+B}(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{4} & ,0 \le x \le 4 \\ -\frac{x}{4}+2 & ,4 \le x \le 8 \\ 0, & x > 8 \end{cases} \quad \ldots(3)$$

Which is represented graphically as shown below



Addition of two Fuzzy number

The substraction of two fuzzy number can be obtained from (1) and (2)

Thus

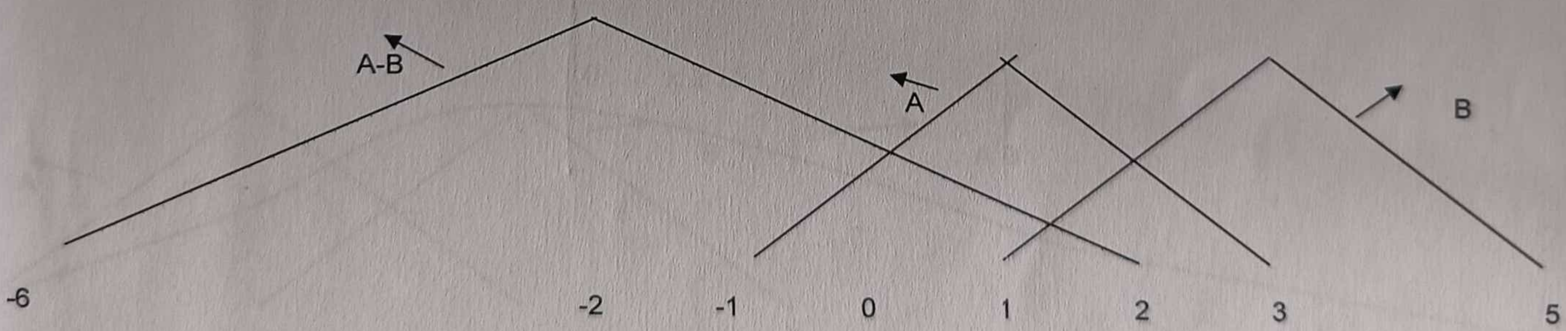
$$A_\alpha(-)B_\alpha = [2\alpha - 1, -2\alpha + 3](-)[2\alpha + 1, -2\alpha + 5]$$

$$= [4\alpha - 6, -4\alpha + 2]$$

Finally

$$\mu_{A(-)B}(x) = \begin{cases} 0 & x \le -6 \\ \dfrac{x+6}{4} & , -6 \le x \le -2 \\ -\dfrac{x}{4} + \dfrac{1}{2} & , -2 \le x \le 2 \end{cases} \quad \ldots(4)$$

Which is represented graphically as shown below



Subtraction of two fuzzy number

Again the interval for multiplication for $A_{(\alpha)}(.)B_{(\alpha)}$ is

$$A_{(\alpha)}(.)B_{(\alpha)} = [2\alpha - 1, -2\alpha + 3](.)[2\alpha + 1, -2\alpha + 5]$$

$$= [(2\alpha - 1)(2\alpha + 1), (-2\alpha + 3)(-2\alpha + 5)]$$

$$= [4\alpha^2 - 1, 4\alpha^2 + 16\alpha - 15]$$

We now have two equations to solve viz

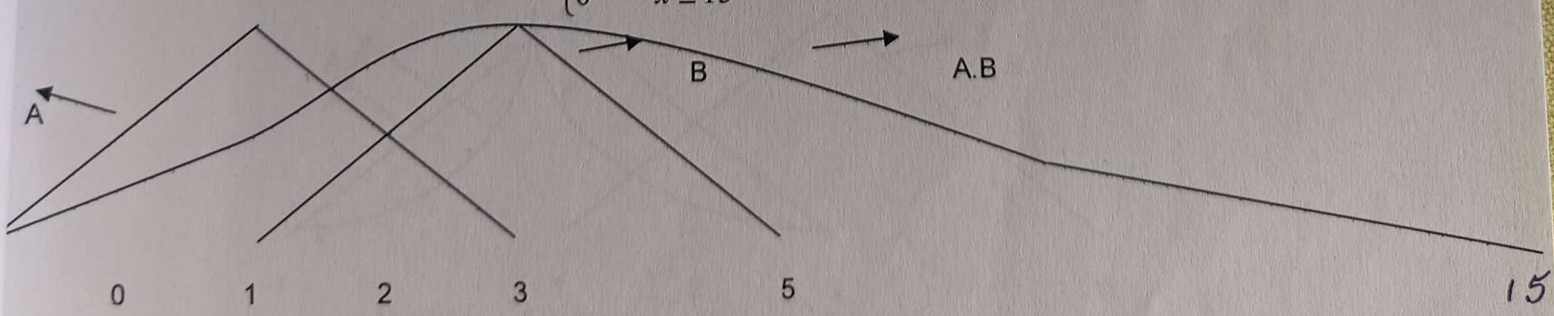
$$4\alpha^2 - 1 - x_1 = 0$$

and

$$4\alpha^2 + 16\alpha - 15 - x_2 = 0$$

Finally

$$\mu_{A.(.)B}(x) = \begin{cases} 0 & , x \leq -1 \\ \frac{1 + \sqrt{17 + 16x}}{8}, & -1 \leq x \leq 3 \\ \frac{16 - \sqrt{16 + 16x}}{8} & , 3 \leq x \leq 15 \\ 0 & x \geq 15 \end{cases} \quad \ldots(5)$$



Multiplication of two fuzzy number

Lastly division of two fuzzy numbers from(1) and (2) would be

$$A_{(\alpha)}(:)B_{(\alpha)} = [2\alpha - 1 \quad , -2\alpha + 3](:)[2\alpha + 1, -2\alpha + 5]$$

$$= \left[\frac{2\alpha - 1}{-2\alpha + 5} \quad , \frac{-2\alpha + 3}{2\alpha + 1}\right]$$

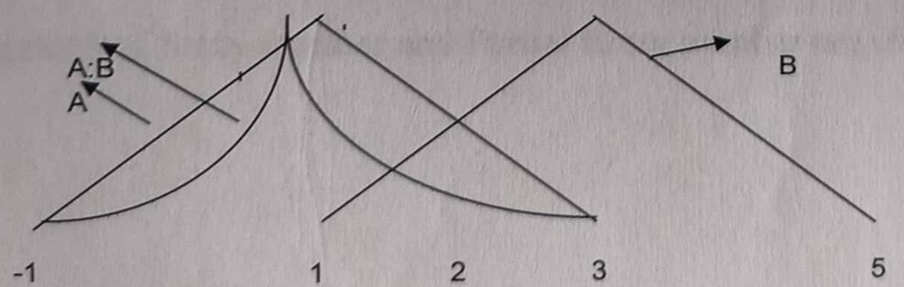
We now have two equations to solve viz,

$$x_1 = \frac{2\alpha - 1}{-2\alpha + 5} \qquad , x_2 = \frac{-2\alpha + 3}{2\alpha + 1}$$

Finally

$$\mu_{A:B}(x) = \begin{cases} 0, \quad x \leq -\frac{1}{5} \\ \frac{5x+1}{2+2x} & , -\frac{1}{5} \leq x \leq \frac{1}{3} \\ \frac{-x+3}{2x+2} & , \frac{1}{3} \leq x \leq 3 \\ 0, x \geq 3 \end{cases}$$

Which can be represented graphically as shown below



Division of two fuzzy number

3.Partial triangular fuzzy number.

If X_1 and X_2 are fuzzy variable and by changing the base line and by taking transformation by rotating the co-ordinate axis ie at α cuts . If X_1 and X_2 are fuzzy variables then

$$Y=X_1-X_2$$

will follow the fuzzy function

$$F(Y) = Y+.84015625-1.95 \quad , \quad 1.95 \leq Y \leq 1.98$$

with $F(1.95)=.84015625$, $F(1.98) = .870156$

At exactness ,these points and F(.) is the particular value of the fuzzy function In the same way if we redefine the fuzzy variabler X_1 and X_2

$$Y=X_1+X_2$$

will follow the fuzzy function

$$F(Y)= Y^2/2-.51984375Y+.01984375 \quad , \quad 1.98 \leq Y \leq 2$$

with $F(1.98)= .911065625$, $F(2) = .98015625$

If we now integrate over Y then if X_1 and X_2 are fuzzy variables then

$$Y=X_1+X_2$$

will follow the fuzzy function

$$F(Y)= Y \quad , \quad 1 \leq Y \leq .9375$$

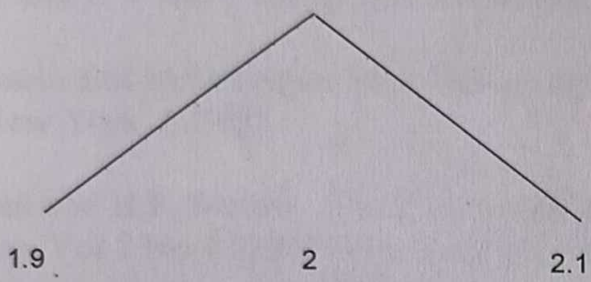
with $F(1)=1$, $F(.9375)=.9375$

Similarly $F(Y)=-Y^2/2 +2.40625Y-.8825$, $.9375 \leq Y \leq .875$

$$F(.9375) = .9330625, \quad F(.875)=.84015625$$

We stop integrating as while integrating over X we have shifted the base line to .84015625 where the two values are same.

The diagrams of fuzzy number and Partial fuzzy number are shown below



Triangular Fuzzy Number

Partial Triangular Fuzzy Number

Partial Fuzzy Number

4.**CONCLUSION**:From the diagrams of addition of two fuzzy number ,subtraction of two fuzzy number,multiplication of two fuzzy number and division of two fuzzy number it is clear that the convergence occur at the Partial fuzzy number in case of addition,subtraction Multiplication and division which is linear and non linear.Moreover we can get exact value or accuracy occurs at the partial fuzzy number which is again an interval.It is normal when its value is 1.Also cluster occurs in the partial fuzzy number.

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An article on Actuarial science

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Abstract

Now-a- days Actuarial science is gaining interest in every sector of insurance. Its origination. Its importance in different disciplines. Its importance in various sectors of insurance business like traditional life insurance, health insurance, pension industry and in social welfare program.

Keywords: Actuarial science, actuaries, annuities, insurance.

1. Introduction.

Actuarial science applies mathematical and statistical methods to finance and insurance, particularly to risk assessment. **Actuaries** are professionals who are qualified in this fields through examinations and experience.

Actuarial science includes a number of interrelating disciplines, including probability and statistics, finance and economics. Historically, actuarial science used deterministic models in the constructions of tables and premiums. The science has gone through revolutionary changes during the last 30 years due to proliferation of high speed computers and the synergy of stochastic actuarial models with modern financial theory.

2. Origination.

Actuarial science became a formal mathematical discipline in the late 17th century with the increased demand for long term insurance coverages such as Burial, Life insurance and Annuities. These long term coverages required that money to be set aside to pay future benefits, such as annuity and death benefits many years into the future. This require estimating future contingent events, such as the rates of mortality by age, as well as the development of mathematical techniques for discounting the value of funds set aside and invested. This led to the development of an important actuarial concepts, refered to as present value of the future sum. Pensions and healthcare emerged in the early 20th century as a result of collective bargaining. Certain aspects of the actuarial methods for discounting pension funds have come under criticism from modern financial economics.

3. Importance in various insurance sector.

Actuarial science has its great importance in the fields like-

a) **In traditional life insurance**, actuarial science focuses on the analysis of mortality , the production of life tables, and the application of compound interest to produce life insurance, annuities and endowment policies. Contemporary life insurance programs have been extended to include credit and mortgage insurance, key man insurance for small businesses, long term care insurance and health savings accounts.

b) **In health insurance**, including insurance provided directly by employers and social insurance, actuarial science focuses on the analysis of rates of disability, morbidity, mortality, fertility and other contingencies. The effects of consumer choice and the geographical distribution of the utilization of medical services and procedures, and the utilization of drugs and therapies, is also of great importance. These factors underlay the development of the Resource-Based Relative Value Scale (RBRVS) at Harvard in a multi-disciplined study. Actuarial science also aids in the design of benefit structures, reimbursement standards, and the effects of proposed government standards on the cost of healthcare.

c) **In the pension industry**, actuarial methods are used to measure the costs of alternative strategies with regard to the design, maintenance or redesign of pension plans. The strategies are generally influenced by collective bargaining; the employer's old, new and foreign competitors; the changing demographics of the workforce; changes in the internal revenue code; changes in the attitude of the internal revenue service regarding the calculation of surpluses; equally importantly, both the short and long term financial and economic trends. It is common with mergers and acquisitions that several pension plans have to be combined or at least administered on an equitable basis. When benefit changes occur, old and new benefit plans have to be blended, satisfying new social demands and various government discrimination test calculations, and providing employees and retirees with understandable choices and transition paths. Benefit plans liabilities have to be properly valued, reflecting both earned benefits for past service and the benefits for future service. Finally, funding schemes have to be developed that are manageable and satisfy the Financial Accounting Standards Board (FASB).

d) **In social welfare programs** , the Office of the Chief Actuary(OCACT), Social Security Administration plans and direct a program of actuarial estimates and analyses relating to SSA-administered retirement, survivors and disability insurance programs and to propose changes in those programs.

It evaluates operations of the Federal Old-Age and Survivors Insurance Trust Fund and the Federal Disability Insurance Trust Fund, conducts studied of program financing, performs actuarial and demographic research on social insurance and related program issues involving mortality, morbidity, fertility, utilization, retirement, disability, survivorship, marriage, unemployment, poverty, old age, families with children, etc., and projects future workloads. In addition, the Office is charged with conducting cost analyses relating to the Supplemental Security Income (SSI) program, a general-revenue financed , means- tested program for low-income aged, blind and disabled people.

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A Few Words on $M/E_k/1$ Queuing Model with Balking and Reneging Customer

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Abstract

In the context of our fast-paced world, customer's impatience in queuing systems is a reality. Such impatience finds reflection through the concepts of balking and reneging. In this paper, we analyze a queuing system with Erlangian service discipline assuming that customers can balk as well as renege. Analysis in steady state is presented.

Keywords: Balking, Impatience, Queuing, Reneging.

1. Introduction.

In the analysis of queues, queuing theorists have model various queuing systems on the basis of their characteristics. Of the various characteristics, one relates to the nature of customer behavior. Balking and reneging are two aspects of customer behavior. By balking, we mean the phenomena of customers arriving to a queuing system and leaving without joining the system. Haight (1957) has provided a rationale which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency, so that a queue of certain length will not be joined, to indifference where a non-zero queue is also joined. Even if a customer does not balk and joins the queuing system, it is quite often the case that customers are not willing to wait indefinitely for service. The phenomenon of customers joining a queuing system and leaving it without service completion is known as reneging. In our modern fast-paced life, customers are hard pressed for time and hence in our day-to-day life reneging can be observed. In spite of the importance of balking and reneging, one does not very often come across papers in queuing literature, which analyzes these two phenomenons.

Reneging can be of two types-viz. reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). A customer can renege only as long as it is in the queue and we call this as reneging of type R_BOS. It cannot renege once it begins receiving service. A common example is the barbershop.

A customer can renege while he is waiting in queue. However once service get started i.e. hair cut begins, the customer cannot leave till hair cutting is over. On the other hand, if customers can renege not only while waiting in queue but also while receiving service, we call such behavior as reneging of type R_EOS. An example is processing or merchandising of perishable goods.

In this paper, we analyze customer impatience in $M/E_k/1$ model. To the best of our knowledge, this has not been attempted. Here it is assumed that customers arriving into the queuing system follows Markovian law with rate λ. There is one server who offers service in k-stages. The service discipline is assumed to be Erlangian with parameter μ and k. We shall further assume that customers who join this queuing system are of balking as well as reneging type. Balking probability is assumed state independent and is taken as 'p'. Each customer individually has a patience or reneging distribution following exp(ν) which commences at the instant the customer joins the system. In this paper, R_BOS is considered only.

The subsequent sections of this paper are structured as follows. Section 2 contains a brief review of the literature. Section 3 contains the steady state analysis. Concluding statements are presented in section 4.

2.Literature Survey.

One of the earliest work on reneging was by Barrer (1957) where he considered deterministic reneging with single server Markovian arrival and service rates. Customers were selected randomly for service.

In his subsequent work, Barrer (1957) also considered deterministic reneging (of both R_BOS and R_EOS type) in a multi server scenario with FCFS discipline. The general method of solution was extended to two related queuing problems. Another early work was by Haight (1959). Ancher and Gafarian (1963) carried out an early work on Markovian reneging with Markovian arrival and service pattern. Ghosal (1963) considered a D/G/1 model with deterministic reneging. Kok and Tijms (1985) considered a single server queuing system where a customer becomes a lost customer when its service has not begun within a fixed time.

Haghighi et al. (1986) considered a Markovian multi server queuing model with balking as well as reneging. Each customer had a balking probability, which was independent of the state of the system. Reneging discipline considered by them was R_BOS. Liu et al. (1987) considered an infinite server Markovian queuing system with reneging of type R_BOS. Customers had a choice of individual service or batch service, batch service being preferred by the customer. Boots and Tijms (1999) considered an M/M/C queue in which a customer leaves the system when its service has not begun within a fixed interval after its arrival. In this paper, they have given the probabilistic proof of 'loss probability', which was expressed in a simple formula involving the waiting time probabilities in the standard M/M/C queue.

Wang et al. (1999) considered the machine repair problem in which failed machines balk with probability (1-b) and renege according to a negative exponential distribution.

Bae et al. (2001) considered an M/G/1 queue with deterministic reneging. They derived the complete formula of the limiting distribution of the virtual waiting time explicitly. Choi et al. (2001) introduced a simple approach for the analysis of the M/M/C queue with a single class of customers and constant patience time by finding simple Markov process. Applying this approach, they analyzed the M/M/1 queue with two classes of customer in which class 1 customer have impatience of constant duration and class 2 customers have no impatience and lower priority than class 1 customers. Performance measures of both M/M/C and M/M/1 queues were discussed. El- Paoumy (2008) also derived the analytical solution of $M^x/M/2/N$ queue for batch arrival system with Markovian reneging. In this paper, the steady state probabilities and some performance measures of effectiveness were derived in explicit forms. Another paper on Markovian reneging was by Yechiali and Altman (2008).

Other attempts at modeling reneging phenomenon include those by Baccelli et al (1984), Martin and Artalejo (1995), Shawky (1997), Choi, Kim and Zhu (2004), and Singh et al (2007), El- Sherbiny (2008) and El-Paoumy and Ismail (2009) etc.

An early work on balking was by Haight (1957). Another work using the concepts of balking and reneging in machine interference queue has been carried out by Al-Seedy and Al-Ibraheem (2001). Liu and Kulkarni (2008) considered an M/PH/1 queue with work load-dependent balking. They assumed that an arriving customer joined the queue and stayed until served if and only if the system workload was less than a fixed level at the time of his arrival.

They also obtained the mean and LST of the busy period in the M/PH/1 queue with workload-dependent balking as a special limiting case of this fluid model. They illustrate the results with the help of numerical examples. Al-Seedly et al. (2009) presented an analysis for the M/M/c queue with balking and reneging. They assumed that arriving customers balked with a fixed probability and reneged according to a negative exponential distribution. The generating function technique was used to obtain the transient solution of system those results in a simple differential equation.

Bae and Kim (2010) considered a G/M/1 queue in which the patience time of the customers is constant. The stationary distribution of the workload of the server, or the virtual waiting time was derived by the level crossing argument. Boxma et al. (2010) considered an M/G/1 queue in which an arriving customer does not enter the system whenever its virtual waiting time i.e. the amount of work seen upon arrival, was larger than a certain random patience time. They determined the busy period distribution for various choices of the patience time distribution.

Some other papers which have considered both balking and reneging are the work by Shawky and El-Paoumy (2000), El- Paoumy (2008), El- Sherbiny (2008), Shawky and El-Paoumy (2008), Pazgal et al. (2008).

3. The System States Analysis.

Let $p_n(t)$ denotes the probability that there are 'n' phases in the system at time 't' under R_BOS. Then we can have $n \geq 0$ number of phases in the following mutually exclusive ways:

1) There are n phases at time't'; there is no arrival, no service and no customer leaving the system during next Δt. The probability is

$$\begin{aligned} &p_n(t)\{1-\lambda(1-p)\Delta t+0(\Delta t)\}\{1-k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ &= p_n(t)\{1-k\mu\Delta t-\lambda(1-p)\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ &= p_n(t)\{1-\nu\Delta t-k\mu\Delta t-\lambda(1-p)\Delta t+0(\Delta t)\} \end{aligned} \tag{3.1}$$

2) There are (n+1) phases at time't', no arrival, one service and no customer leaving the system during next Δt. The probability is

$$\begin{aligned} &p_{n+1}(t)\{1-\lambda(1-p)\Delta t+0(\Delta t)\}\{k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ &= p_{n+1}(t)\{k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ &= p_{n+1}(t)\{k\mu\Delta t+0(\Delta t)\} \end{aligned} \tag{3.2}$$

3) There are (n-k) phases at time 't', one arrival, no service and no customer leaving the system during the next Δt. The probability is

$$\begin{aligned} &p_{n-k}(t)\{\lambda(1-p)\Delta t+0(\Delta t)\}\{1-k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ &= p_{n-k}(t)\{\lambda(1-p)\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ &= p_{n-k}(t)\{\lambda(1-p)\Delta t+0(\Delta t)\} \end{aligned} \tag{3.3}$$

4) There are (n+k) phases at time 't', no arrival, no service and one customer leaving the system during the next Δt. The probability is

$$\begin{aligned} &p_{n+k}(t)\{\lambda(1-p)\Delta t+0(\Delta t)\}\{1-k\mu\Delta t+0(\Delta t)\}\{\nu\Delta t+0(\Delta t)\} \\ &= p_{n+k}(t)\{\lambda(1-p)\Delta t-k\mu\Delta t+0(\Delta t)\}\{\nu\Delta t+0(\Delta t)\} \\ &= p_{n+k}(t)\{\nu\Delta t+0(\Delta t)\} \end{aligned} \tag{3.4}$$

From (3.1), (3.2), (3.3) and (3.4) we have,

$$p_n(t+\Delta t) = p_n(t)\{1 - v\Delta t - \lambda(1-p)\Delta t - k\mu\Delta t + 0(\Delta t)\} + k\mu\Delta t p_{n+1}(t) + 0(\Delta t)p_{n+1}(t)$$
$$+ p_{n-k}(t)\{\lambda(1-p)\Delta t + 0(\Delta t) + p_{n+k}(t)\{v\Delta t + 0(\Delta t)\}$$
$$\Rightarrow p_n(t+\Delta t) - p_n(t) = -\{v + \lambda(1-p) + k\mu\}\Delta t p_n(t) + 0(\Delta t)p_n(t) + k\mu\Delta t p_{n+1}(t) + 0(\Delta t)p_{n+1}(t) +$$
$$\lambda(1-p)\Delta t p_{n-k}(t) + 0(\Delta t)p_{n-k}(t) + v\Delta t p_{n+k}(t) + 0(\Delta t)p_{n+k}(t)$$

Dividing both sides of this equation by Δt and taking limit Δt→0, we get

$$p_n'(t) = -\{\lambda(1-p) + v + k\mu\}p_n(t) + k\mu p_{n+1}(t) + \lambda(1-p)p_{n-k}(t) + vp_{n+k}(t)\;;n\geq 1 \qquad (3.5)$$

Again, under R_BOS, we have 0 phases at time (t+Δt) in the following mutually exclusive ways:

1) 0 phases at time 't', no arrival, no service and no customer leaving the system during next Δt. The probability is

$$p_0(t)\{1 - \lambda\Delta t + 0(\Delta t)\} \qquad (3.6)$$

2) 1 phase at time 't', no arrival, one service and no customer leaving the system during next Δt. The probability is

$$p_1(t)\{1 - \lambda(1-p)\Delta t + 0(\Delta t)\}\{k\mu\Delta t + 0(\Delta t)\}\{1 - v\Delta t + 0(\Delta t)\}$$
$$= p_1(t)\{k\mu\Delta t + 0(\Delta t)\}\{1 - v\Delta t + 0(\Delta t)\} \qquad (3.7)$$
$$= p_1(t)\{k\mu\Delta t + 0(\Delta t)\}$$

From (3.6) and (3.7) we have ,

$$p_0(t+\Delta t) = p_0(t)\{1 - \lambda\Delta t + 0(\Delta t)\} + p_1(t)\{k\mu\Delta t + 0(\Delta t)\}$$
$$\Rightarrow p_0(t+\Delta t) - p_0(t) = -\lambda\Delta t p_0(t) + p_0(t)0(\Delta t) + k\mu\Delta t p_1(t) + 0(\Delta t)p_1(t)$$

Now dividing both sides of this equation by Δt and taking limit $\Delta t \to 0$, we get

$$p_0'(t) = -\lambda p_0(t) + k\mu p_1(t)$$

(3.8)

Under R_BOS and steady state, the differential equations (3.5) and (3.8) becomes

$$\lambda(1-p)p_{n-k} + k\mu p_{n+1} + \nu p_{n+k} = \{\lambda(1-p) + \nu + k\mu\} p_n$$

(3.9)

$$\lambda p_0 = k\mu p_1$$

(3.10)

Now multiplying both sides of equation (3.9) by z^n and summing over n and after doing some algebra we get,

$$P(z) = [\{\nu z + k\mu(z-1) - \lambda p z\} p_0 - \nu z \sum_{n=0}^{k} p_n z^n] \Big/ \{\lambda z(1-p) + k\mu(z-1) - \lambda(1-p) z^{k+1}\}$$

(3.11)

Putting p=0 and ν=0 in (3.11) we can obtain the probability generating function of traditional $M/E_k/1$ model. (Medhi, 2003, pp: 165-166).

4. Conclusion:

The analysis of a Erlangian queuing system with balking and Markovian reneging has been presented. Analysis in steady state has been discussed.

Even though balking and reneging have been discussed by others, explicit expression are not available. This paper makes a contribution here. Particular case has been attempted. The limitations of this work stem from the Markovian assumptions of arrival pattern. Extension of our results for general distribution is a pointer to future research.

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THE POPULATION MODEL WITH RELATIVE GROWTH RATE USING FUZZY DATA

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ABSTRACT

In our work we have used the model with hyperbolic relative growth rate derived by Chakravarty and Baruah (1992) with fuzzy data. We shall also derive a fuzzy model to describe the number of persons of a geographical region at any age which was derived on the assumption that the RGR of the population concerned at any age decreases hyperbolically over time and also see the effect of fuzziness on the model of hyperbolic growth rate. The fuzzy model developed will be applied to the situation prevailing in India as India is with large population.

Key Words: fuzzy membership function, fuzzy model of hyperbolic relative growth rate

1. INTRODUCTION

The type of growth has always been a subject of research since the very inception of civilization. Research of the growth of human population was first started by Thomas R. Malthus followed by Lotka .The deterministic model of population dynamic due to Lotka is based on the model due to Malthus represented by a first ordering differential equation involving total population dependent on time. Many population mathematicians especially Pearl and Rhodes have modified the Malthusian differential equation to obtain an equation of logistic curve.

The Logistic law of population growth is based on the assumption that the relative growth rate (RGR) of the population under considering decreases linearly as the population size increases and population cannot grow beyond a limit. The size of the population defined by the logistic law approaches a certain maximum but never attains it. Under the assumption that the RGR of the population concerned decreases hyperbolically over time, a model was developed by Chakravarty and Baruah (1990). In this model it is assumed that the size of the population under consideration attains certain maximum. Also the assumption that the population cannot grow beyond a finite limit is retained here.

In a third world country like India, the data on population may not actually be very precise. Hence the concept of fuzzification can be taken into consideration. In the fuzzy model of population growth rate can be constructed representing the observed population expressed in thousand as fuzzy. Our objective is to see the effect of fuzziness on the model in finding the projected population.

2. MODEL DESCRIBING TOTAL POPULATION

Let $P(t)$ be the size of the population of a geographical region at time t, then at time t the relative rate of growth, relative to the total population is –

$$\lim_{\Delta t \to 0+} \frac{P(t+\Delta t)-P(t)}{P(t)\Delta t} = \frac{1}{P(t)} \cdot \frac{d}{dt}P(t) \tag{2.1}$$

The RGR can neither be constant nor be an increasing function of time (Rhodes 1940) for it is so, the population would increase without limit which is unrealistic .So it is reasonable to consider the RGR as a decreasing function of time. It is assumed that it decreases hyperbolically in time so that one can consider the form

$$\frac{1}{P} \cdot \frac{dP}{dt} = a + \frac{b}{t}, \quad b>0 \text{ and } P = P(t) \tag{2.2}$$

However the above is not defined for the origin .For the RGR to exist at the origin (2.2) is modified to the form

$$\frac{1}{P} \cdot \frac{dP}{dt} = a + \frac{b}{t+T_0}, \; T_0 > 0 \tag{2.3}$$

where T_0 is to be chosen arbitrarily. No population can grow without limit due to the limitation of resources to support the population. Accordingly the population must have a limiting size .Let L be the limiting size of the population . Further it is assumed that the population attains its maximum at time T. Then at $t = T$, the RGR of the population must vanish so the equation (2.3) yields-

$$b = -a(T+T_0) \tag{2.4}$$

We therefore have

$$\frac{1}{P} \cdot \frac{dP}{dt} = a.(1 - \frac{T+T_0}{t+T_0}) \tag{2.5}$$

This is the form of the RGR of the population concerned a and T are two parameter here.

Integrating he differential equation (2.5) we get-

$$\log P = a[t-(T+T_0).\log(t+T_0)] + c \tag{2.6}$$

c being the constant of integration. These yields –

$$P(t) = Ae^{at}.(t+T_0)^{-a(T+T_0)} \tag{2.7}$$

i.e $P(t) = \exp\{\ln A + aT - a(T+T_0)\ln(t+T_0)\}$ (2.8)

A being some parameter.

Therefore we have

$$P(t) = P_0 e^{at}.(1+\frac{t}{T_0})^{-a(T+T_0)} \tag{2.9}$$

This is a model to describe the total population of the geographical region.

From equation (2.7) by using the recurrence relation the equation for the projected population can be obtained, which is given by –

$$P(t+\eta) = e^{\eta a}(1+\frac{\eta}{t+T_0})^{-a(T+T_0)} \tag{2.10}$$

i.e. $P(t+\eta) = \exp\{\eta a - a(T+T_0)\ln(1+\frac{\eta}{t+T_0})\}$ (2.11)

3. MODEL DESCRIBING TOTAL POPULATION USING FUZZY DATA

It P (t) be the size of the population of a geographical region at time t then from the equation (2.8) the fuzzy model describing the total population will be

$$P(t)\,(=)\exp\{\ln \hat{A}(+)at(-)a(T+T_0)\ln(t+T_0)\} \tag{3.1}$$

From equation (2.11) the projected population will be-

$$P(t+n)\,(=)\exp\{\,na(-)a(T+T_0)\ln(1+\frac{n}{t+T_0})\}$$

For our convenience we will use the following equation which is given by

$$P(t+\eta)\,(=)\exp\{\,\eta a(-)a(T+T_0)\ln(1+\frac{\eta}{t+T_0})\} \tag{3.2}$$

4. THE ORIGINAL METHOD OF FITTING THE MODEL

To fit the model developed to a set of observed data Chakravarty and Baruah (1992) have used two methods.

As there are three parameters in this model three equispaced observations are required. In this method it is assumed that $P(t_0), P(t_1)$ and $P(t_2)$ be three observations that more or less cover the whole set of the given observations with $t_1-t_0=t_2-t_1$. By suitable choice of origin and scale t_0, t_1 and t_2 is made corresponding to 0, 1, 2 respectively so that $P(t_0), P(t_1)$ and $P(t_2)$ become $P(0), P(1)$ and $P(2)$ respectively. Thus from the equation (2.8) gives –

$$\ln P(t)=\ln A+at-a(T+T_0)\ln(t+T_0) \tag{4.1}$$

This gives

$$\left.\begin{aligned}\ln P(0)&=\ln A-a\cdot(T+T_0)\ln T_0\\ \ln P(1)&=\ln A+a+a\cdot(T+T_0)\ln(1+T_0)\\ \ln P(2)&=\ln A+2a-a\cdot(T+T_0)\ln(2+T_0)\end{aligned}\right\} \tag{4.2}$$

Accordingly

$$\left.\begin{aligned}\Delta\ln P(0)&=a-a(T+T_0)\ln(\frac{1+T_0}{T_0})\\ \Delta\ln P(1)&=a-a(T+T_0)\ln(\frac{2+T_0}{1+T_0})\end{aligned}\right\} \tag{4.3}$$

Finally

$$\Delta^2\ln P(0)=-\,a(T+T_0)\ln\left\{\frac{T_0(2+T)}{(1+T_0)^2}\right\} \tag{4.4}$$

5. THE METHOD USING FUZZY DATA

Let $P(t_0), P(t_1)$ and $P(t_2)$ be three fuzzy observations with $t_1-t_0=t_2-t_1$. By suitable choice of origin and scale t_0, t_1 and t_2 can be made corresponding to 0, 1, 2 respectively so that $P(t_0), P(t_1)$ and $P(t_2)$ become $P(0), P(1)$ and $P(2)$ respectively. Thus from 4(A) the fuzzy equations will be –

$$\ln P(t)\,(=)\ln A(+)at(-)a(T+T_0)\ln(t+T_0) \quad (5.1)$$

This gives

$$\left.\begin{aligned} &\ln P(0)\,(=)\ln A(-)\,a(\cdot)(T+T_0)\ln T_0 \\ &\ln P(1)\,(=)\ln A\,(+)\,a\,(+)\,a(\cdot)(T+T_0)\ln(1+T_0) \\ &\ln P(2)\,(=)\ln A\,(+)\,2a\,(-)\,a(\cdot)(T+T_0)\ln(2+T_0) \end{aligned}\right\} \quad (5.2)$$

Accordingly

$$\left.\begin{aligned} &\Delta \ln P(0)\,(=)\,a(-)\,a(T+T_0)\ln(\frac{1+T_0}{T_0}) \\ &\Delta \ln P(1)\,(=)\,a(-)\,a(T+T_0)\ln(\frac{2+T_0}{1+T_0}) \end{aligned}\right\} \quad (5.3)$$

Finally

$$\Delta^2 \ln P(0)\,(=)-a(T+T_0)\ln\left\{\frac{T_0(2+T)}{(1+T_0)^2}\right\} \quad (5.4)$$

6. THE FUZZY ARITHMATIC CONCERNED-

Let $P(0), P(1)$ and $P(2)$ be the fuzzy population for the year t_0, t_1 and t_2. Then we have

$P(0)\,(=)[P(0_1), P(0_2), P(0_3)]$, $P(1)\,(=)[P(1_1), (1_2), P(1_3)]$ and $P(2)\,(=)[P(2_1), (2_2), P(2_3)]$ such that $P(0_2)-P(0_1)=P(0_3)-P(0_2)$, $P(1_2)-P(1_1)=P(1_3)-P(1_2)$ and $P(2_2)-P(2_1)=P(2_3)-P(2_2)$

Thus we have

$$\ln P(0)\,(=)[\ln P(0_1), \ln P(0_2), \ln P(0_3)] \quad (6.1)$$

$$\ln P(1)\,(=)[\ln P(1_1), \ln P(1_2), \ln P(1_3)] \quad (6.2)$$

$$\ln P(2)\,(=)[\ln P(2_1), \ln P(2_2), \ln P(2_3)] \quad (6.3)$$

α cut of $\ln P(0)$ is

$$\ln P(0)^{(\alpha)}\,(=)[\ \ln\frac{P(0_2)}{P(0_1)}\alpha + \ln P(0_1),\ -\ln\frac{P(0_3)}{P(0_2)}\alpha + \ln P(0_3)]$$

$$x_1 = \ln\frac{P(0_2)}{P(0_1)}\alpha + \ln P(0_1) \quad (6.4)$$

$$x_2 = -\ln\frac{P(0_3)}{P(0_2)}\alpha + \ln P(0_3) \quad (6.5)$$

Solving for α from the above two equations we get the fuzzy membership function (f.m.f) of $\ln P(0)$. Thus the f.m.f of $\ln P(0)$ is -

$$\begin{aligned} \mu_{\ln P(0)}(x) &= \frac{x-\ln P(0_1)}{\ln\frac{P(0_2)}{P(0_1)}}, \quad \ln P(0_1)\le x\le \ln P(0_2) \\ &= \frac{-x+\ln P(0_3)}{\ln\frac{P(0_3)}{P(0_2)}}, \quad \ln P(0_2)\le x\le \ln P(0_3) \\ &= 0,\ \text{otherwise} \end{aligned} \quad (6.6)$$

Similarly we have obtained the α cut and f.m.f of $\ln P(1)$ and $\ln P(2)$.

Putting the α cut of $\ln P(0)$ and $\ln P(1)$ in $\Delta \ln P(0)(=) \ln P(1)(-)P(0)$ we have the α cut of $\ln P(0)$ as-

$$\Delta \ln P(0)(=)[\ln \frac{P(0_3)P(1_2)}{P(0_2)P(1_1)}\alpha + \ln \frac{P(1_1)}{P(0_3)}, -\ln \frac{P(0_2)P(1_3)}{P(0_1)P(1_2)}\alpha + \ln \frac{P(1_3)}{P(0_1)}$$

f.m.f of $\Delta \ln P(0)$ is

$$\mu_{\Delta \ln P(0)}(x) = \frac{x - \ln P\frac{P(1_1)}{P(0_3)}}{\ln \frac{P(0_3)P(1_2)}{P(0_2)P(1_1)}}, \quad \ln \frac{P(1_1)}{P(0_3)} \le x \le \ln \frac{P(1_2)}{P(0_2)}$$

$$= \frac{-x + \ln P\frac{P(1_3)}{P(0_1)}}{\ln \frac{P(0_2)P(1_3)}{P(0_1)P(1_2)}}, \quad \ln \frac{P(1_2)}{P(0_2)} \le x \le \ln \frac{P(1_3)}{P(0_1)}$$

$$= 0, \text{ otherwise} \qquad (6.7)$$

Putting the α cut of $\ln P(1)$ and $\ln P(2)$ in $\Delta \ln P(1)(=) \ln P(2)(-)P(1)$, we have the α cut of $\ln P(1)$ as-

$$\Delta \ln P(1)^{(\alpha)} (=) \ [\ln \frac{P(1_3)P(2_2)}{P(1_2)P(2_1)}\alpha + \ln \frac{P(2_1)}{P(1_3)}, -\ln \frac{P(1_2)P(2_3)}{P(1_1)P(2_2)}\alpha + \ln \frac{P(2_3)}{P(1_1)}]$$

f.m.f of $\Delta \ln P(1)$ is

$$\mu_{\Delta \ln P(1)}(x) = \frac{x - \ln P\frac{P(2_1)}{P(1_3)}}{\ln \frac{P(1_3)P(2_2)}{P(1_2)P(2_1)}}, \quad \ln \frac{P(2_1)}{P(1_3)} \le x \le \ln \frac{P(2_2)}{P(1_2)}$$

$$= \frac{-x + \ln \frac{P(2_3)}{P(1_1)}}{\ln \frac{P(1_2)P(2_3)}{P(1_1)P(2_2)}}, \quad \ln \frac{P(2_2)}{P(1_2)} \le x \le \ln \frac{P(2_3)}{P(1_1)}$$

$$= 0, \text{ otherwise} \qquad (6.8)$$

Putting the α cut of $\Delta \ln P(0)$ and $\Delta \ln P(1)$ in $\Delta^2 \ln P(0)(=)\Delta \ln P(1)(-)\Delta P(0)$ is

$$\Delta^2 \ln P(0) (=) [\ \ln \frac{P(0_2)P(1_3)^2 P(2_2)}{P(0_1)P(1_2)^2 P(2_1)}\alpha + \ln \frac{P(0_1)P(2_1)}{P(1_3)^2}, -\ln \frac{P(0_3)P(1_2)^2 P(2_3)}{P(0_2)P(1_1)^2 P(2_2)}\alpha + \ln \frac{P(0_3)P(2_3)}{P(1_1)^2}$$

Thus f.m.f of $\Delta^2 \ln P(0)$ is

$$\mu_{\Delta^2 \ln P(0)}(x) = \frac{x - \ln \frac{P(0_1)P(2_1)}{P(1_3)^2}}{\ln \frac{P(0_2)P(1_3)^2 P(2_2)}{P(0_1)P(1_2)^2 P(2_1)}}, \quad \ln \frac{P(0_1)P(2_1)}{P(1_3)^2} \le x \le \ln \frac{P(0_2)P(2_2)}{P(1_2)^2}$$

$$= \frac{-x + \ln \frac{P(0_3)P(2_3)}{P(1_1)^2}}{\ln \frac{P(0_3)P(1_2)^2 P(2_3)}{P(0_2)P(1_1)^2 P(2_2)}}, \quad \ln \frac{P(0_2)P(2_2)}{P(1_2)^2} \le x \le \ln \frac{P(0_3)P(2_3)}{P(1_1)^2}$$

$$= 0, \text{ otherwise} \qquad (6.9)$$

Here T_0 is selected as 1. Solving three equations (5.2), (5.3) and (5.4) we obtain the α cut and f.m.f of the estimates which are given below.

The α cut of $\hat{B}(=)-a(T+T_0)$ is

$$\hat{B}^{(\alpha)}(=)[3.47606\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}\alpha+3.47606\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2},$$

$$-3.47606\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}\alpha+3.47606\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}]$$

Thus f.m.f of is $\hat{B}$ is

$$\mu_{\hat{B}}(x)=$$

$$\frac{x-3.47606\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}}{3.47606\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}},3.47606\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}\le x\le 3.47606\ln\frac{P(1_2)^2}{P(0_2)^2P(2_2)^2}$$

$$=\frac{-x+3.47606\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}}{3.47606\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}},3.47606\ln\frac{P(1_2)^2}{P(0_2)^2P(2_2)^2}\le x\le 3.47606\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}$$

$$= 0 \text{ , otherwise} \tag{6.10}$$

The α cut of $\hat{a}$ is

$$\hat{a}^{(\alpha)}(=)$$

$$[\{\ln\frac{P(0_3)P(1_2)}{P(0_2)P(1_1)}+2.40942\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}\}\alpha+\ln\frac{P(1_1)}{P(0_3)}+2.40942\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)},$$

$$-\{\ln\frac{P(0_2)P(1_3)}{P(0_1)P(1_2)}+2.40942\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_1)P(1_1)^4P(2_1)P(2_2)}\}\alpha+\ln\frac{P(1_3)}{P(0_1)}+2.40942\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}]$$

Thus the f.m.f of $\hat{a}$ is,

$$\mu_{\hat{a}}(x)=\frac{x-\ln\frac{P(1_1)}{P(0_3)}-2.40942\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}}{\ln\frac{P(0_3)P(1_2)}{P(0_2)P(1_1)}+2.40942\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}},$$

$$\text{if } \ln\frac{P(1_1)}{P(0_3)}+2.40942\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}\le x\le \ln\frac{P(1_2)}{P(0_2)}+2.40942\ln\frac{P(0_2)P(2_2)}{P(1_2)^2}$$

$$=\frac{-x+\ln\frac{P(1_3)}{P(0_1)}-2.40942\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}}{\ln\frac{P(0_2)P(1_3)}{P(0_1)P(1_2)}+2.40942\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}},$$

$$\text{if, } \ln\frac{P(1_2)}{P(0_2)}+2.40942\ln\frac{P(0_2)P(2_2)}{P(1_2)^2}\le x\le \ln\frac{P(1_3)}{P(0_1)}+2.40942\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}$$

$= \quad 0 \quad , \quad$ otherwise (6.11)

The α cut of $\hat{A}$ is

$$\hat{A}^{(\alpha)}\ (=)[\exp\{\ln\frac{P(0_2)}{P(0_1)}\alpha+\ln P(0_1)\},\exp\{-\ln\frac{P(0_3)}{P(0_2)}\alpha+\ln P(0_3)\}]$$

Thus the f.m.f of $\hat{A}$ is

$$\mu_{\hat{A}}(x) = \frac{\ln x-\ln P(0_1)}{\ln\frac{P(0_2)}{P(0_1)}}\ , \quad P(0_1)\le x\le P(0_2)$$

$$= \frac{-\ln x+\ln P(0_3)}{\ln\frac{P(0_3)}{P(0_2)}}\ , \quad P(0_2)\le x\le P(0_3)$$

$$= \quad 0 \quad , \quad \text{otherwise}$$

(6.12)

From equation (3.1) α cut for estimated population $P(\hat{t})$ is-

$P(\hat{t})^{(\alpha)}(=)$

$$[\exp\{\{\ln\frac{P(0_2)P(0_3)^t P(1_2)^t}{P(0_1)P(0_2)^t P(1_1)^t}+2.40942t\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}+3.47606\ln(t+T_0)\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(}$$

$$+\ln\frac{P(0_1)P(1_1)^t}{P(0_3)^t}+2.40942t\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}+3.47606\ln(t+T_0)\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}\},\exp\{-\{\ln\frac{P(0_2)^tP(0_3)P(}{P(0_1)^tP(0_2)P}$$

$$2.40942t\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}+3.47606\ln(t+T_0)\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}\}\alpha+\ln\frac{P(0_3)P(1_3)^t}{P(0_1)^t}$$

$$2.40942t\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}+3.47606\ln(t+T_0)\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}\}]$$

Thus, f.m.f of $P(\hat{t})$ is

$\mu_{P(\hat{t})}(x)$

$=$

$$\frac{\ln x-\ln\frac{P(0_1)P(1_1)^t}{P(0_3)^t}-2.40942t\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}-3.47606\ln(t+T_0)\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}}{\ln\frac{P(0_2)P(0_3)^tP(1_2)^t}{P(0_1)P(0_2)^tP(1_1)^t}+2.40942t\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}+3.47606\ln(t+T_0)\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}}$$

If, $\exp\{\ln+2.40942t\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}+3.47606\ln(t+T_0)\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}\}$

$\le x\le\exp\{\ln\frac{P(0_2)P(1_2)^t}{P(0_2)^t}+2.40942t\ln\frac{P(0_2)^2P(2_2)^2}{P(1_2)^2}+3.47606\ln(t+T_0)\ln\frac{P(1_2)^2}{P(0_2)P(2_2)}\}$

$=$

$$\frac{-\ln x+\ln\frac{P(0_3)P(1_3)^t}{P(0_1)^t}-2.40942\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}+3.47606\ln(t+T_0)\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}}{\ln\frac{P(0_2)^tP(0_3)P(1_3)^t}{P(0_1)^tP(0_2)P(1_2)^t}+2.40942\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)^2P(1_1)^4P(2_1)P(2_2)}+3.47606\ln(t+T_0)\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}}$$

If, $\exp\{\ln\frac{P(0_2)P(1_2)^t}{P(0_2)^t}+2.40942t\ln\frac{P(0_2)^2P(2_2)^2}{P(1_2)^2}+3.47606\ln(t+T_0)\ln\frac{P(1_2)^2}{P(0_2)P(2_2)}\}$

$\leq x\leq\exp\{\ln\frac{P(0_3)P(1_3)^t}{P(0_1)^t}+2.40942\ln\frac{P(0_3)^2P(1_{31})^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}+3.47606\ln(t+T_0)\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}\}$

$=\qquad 0 \qquad , \qquad$ otherwise

(6.13)

Using the equation (3.2) the α cut for projected population $P(t+\eta)$ is

$P(t+\eta)^{(\alpha)}$ (=)

$[\exp\{\{\ln\frac{P(t_2)P(0_3)^\eta P(1_2)^\eta}{P(t_1)P(0_2)^\eta P(1_1)^\eta}+2.40942\eta\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}+Q(t,\eta)3.47606\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2}$ [illegible]

$+\ln\frac{P(t_1)P(1_1)^\eta}{P(0_3)^\eta}+2.40942\eta\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}+Q(t,\eta)3.47606\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}\}$, $\exp\{-\{\ln\frac{P(t_3)P(0_2)^\eta P(}{P(t_2)P(0_1)^\eta P(1}$ [illegible]

$2.40942\eta\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}+Q(t,\eta)3.47606\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}\}\alpha+\ln\frac{P(t_3)P(1_3)^\eta}{P(0_1)^\eta}+$

$2.40942\eta\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}+Q(t,\eta)3.47606\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}\}]$

where $Q(t,\eta)=\ln(1+\frac{\eta}{t+T_0})$

Thus the f.m.f of $P(t+\eta)$ is –

$\mu_{P(t+\eta)}(x)=$

$$\frac{\ln x-\{\ln\frac{P(0_1)P(1_1)^\eta}{P(0_3)^\eta}+2.40942\eta\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}+Q(t,\eta)3.47606\ln(t+T_0)\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}\}}{\ln\frac{P(t_2)P(0_3)^\eta P(1_2)^\eta}{P(t_1)P(0_2)^\eta P(1_1)^\eta}+2.40942\eta\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}+Q(t,\eta)3.47606\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}}$$

If,

$\exp\{\ln\frac{P(0_1)P(1_1)^\eta}{P(0_3)^\eta}+2.40942\eta\ln\frac{P(0_1)^2P(1_1)^2P(2_1)^2}{P(0_3)P(1_3)^4P(2_3)}+Q(t,\eta)3.47606\ln(t+T_0)\ln\frac{P(0_1)P(1_1)^4P(2_1)}{P(0_3)^2P(1_3)^2P(2_3)^2}\}$

$\leq x\leq\exp\{\ln\frac{P(t_2)P(1_2)^\eta}{P(0_2)^\eta}+2.40942\eta\ln\frac{P(0_2)P(2_2)}{P(1_2)^2}+Q(t,\eta)3.47606\ln\frac{P(1_2)^2}{P(0_2)P(2_2)}\}$

$$\frac{-\ln x+\ln\frac{P(0_3)P(1_3)^{\eta}}{P(0_1)^{\eta}}+2.40942\eta\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}+Q(t,\eta)3.47606\ln(t+T_0)\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}}{\ln\frac{P(t_3)P(0_2)^{\eta}P(1_3)^{\eta}}{P(t_2)P(0_1)^{\eta}P(1_2)^{\eta}}+2.40942\eta\ln\frac{P(0_3)^2P(1_2)^2P(1_3)^2P(2_3)^2}{P(0_1)P(0_2)P(1_1)^4P(2_1)P(2_2)}+Q(t,\eta)3.47606\ln\frac{P(0_2)P(0_3)P(1_3)^4P(2_2)P(2_3)}{P(0_1)^2P(1_1)^2P(1_2)^2P(2_1)^2}}$$

If,

$$\exp\{\ln\frac{P(t_2)P(1_2)^{\eta}}{P(0_2)^{\eta}}+2.40942\eta\ln\frac{P(0_2)P(2_2)}{P(1_2)^2}+Q(t,\eta)3.47606\ln\frac{P(1_2)^2}{P(0_2)P(2_2)}\}\le x\le$$

$$\exp\{\ln\frac{P(0_3)P(1_3)^{\eta}}{P(0_1)^{\eta}}+2.40942\eta\ln\frac{P(0_3)^2P(1_3)^2P(2_3)^2}{P(0_1)P(1_1)^4P(2_1)}+Q(t,\eta)3.47606\ln(t+T_0)\ln\frac{P(0_3)P(1_3)^4P(2_3)}{P(0_1)^2P(1_1)^2P(2_1)^2}\}$$

$= 0$, otherwise

(6.14)

6.1 NUMERICAL EXAMPLE:

We have taken data from the census held in India since 1941 to2001. Here table 1.1 shows the observed population of India for interval of ten years.

Table 1.1

(Observed total population of India)

Year	Population
1941	318660580
1951	361088090
1961	439234771
1971	548159652
1981	683329097
1991	846421039
2001	1028737436

Applying the equation (4.1), (4.2), (4.3) and (4.4) we have obtained the following estimates

$\hat{B} = 0.11412449$

$\hat{a} = 0.36283746$

$\hat{A} = 439234771$

The estimated and the projected total population obtained by applying equation (2.7) and (2.10) and using these estimates have been given below –

Table 1.2

(Estimated total population of India)

Year	Population
1961	439234771
1971	551546573
1981	683329097
1991	840386124
2001	1028737440

Table 1.3

(Projected total population of India)

Year	Population
2011	1255260798
2016	1385313543
2021	1528063565

Here population of 1961 is taken as $P(0)$, 1981 as $P(1)$ and 2001 as $P(2)$. Hence the observed population in fuzzy are given in Table 1.4

Table 1.4

(Observed population in fuzzy)

year	t	$P(t)$	$\ln P(t)$
1961	0	[439.22,439.23,439.24]	[6.085000427,6.085023194,6.085045961]
1981	1	[683.32,683.33,683.34]	[6.526963271,6.526977905,6.52699254]
2001	2	[1028.73,1028.74,1028.75]	[6.936080311,6.936090031,6.936099752]

From equation (6.6) the α cut of $\ln P(0), \ln P(1)$ and $\ln P(2)$ is given below -

$\ln P(0)^{(\alpha)} (=) [0.00002\alpha + 6.08500, -0.00003\alpha + 6.08505]$

$\ln P(1)^{(\alpha)} (=) [0.00002\alpha + 6.52696, -0.00001\alpha + 6.52699]$

$\ln P(2)^{(\alpha)} (=) [0.00001\alpha + 6.93608, -0.00001\alpha + 6.93610]$

From equation (6.7) the α cut of $\Delta \log P(0)$ is

$\Delta \log P(0)^{(\alpha)} (=) [0.00005\alpha + 0.44191, -0.00003\alpha + 0.44199]$

f.m.f of $\Delta \log P(0)$ is

$$\begin{aligned}\mu_{\Delta \log P(0)} &= \frac{x-0.44191}{0.00005}, \quad 0.44191 \le x \le 0.44196 \\ &= \frac{-x+0.44199}{0.00003}, \quad 0.44199 \le x \le 0.44199 \\ &= 0, \text{ otherwise}\end{aligned} \qquad (6.1.1)$$

From equation (6.8) the α cut of $\Delta \log P(1)$ is

$\Delta \log P(1)^{(\alpha)} (=) [0.00002\alpha + 0.40909, -0.00003\alpha + 0.40914]$

f.m.f of $\Delta \log P(1)$ is

$$\begin{aligned}\mu_{\Delta \log P(1)} &= \frac{x-0.40909}{0.00002}, \quad 0.40909 \le x \le 0.40911 \\ &= \frac{-x+0.40914}{0.00003}, \quad 0.40911 \le x \le 0.40914 \\ &= 0, \quad \text{otherwise}\end{aligned} \qquad (6.1.2)$$

From equation (6.9) the α cut of $\Delta^2 \log P(0)$ is

$\Delta^2 \log P(0)^{(\alpha)} (=) [0.00005\alpha - 0.0329, -0.00008\alpha - 0.03277]$

Thus f.m.f of $\Delta^2 \log P(0)$ is

$$\mu_{\Delta^2 \log P(0)} = \frac{x+0.0329}{0.00005} \quad , \text{-}0.0329 \le x \le -0.03285$$
$$= \frac{-x-0.03277}{0.00008} \quad , \text{-}0.03285 \le x \le -0.03277$$
$$= 0, \text{ otherwise} \qquad (6.1.3)$$

From equation (6.10) the α cut of $\hat{B}$ is

$\hat{B}^{(\alpha)}$ (=)[$0.00073\alpha + 0.11345, \text{-}0.00063\alpha + 0.11481$]

Thus its f.m.f is

$$\mu_{\hat{B}}(x) = \frac{x+0.11345}{0.00073} \quad , 0.11345 \le x \le 0.11418$$
$$= \frac{-x+0.11481}{0.00063} \quad , 0.11418 \le x \le 0.11481$$
$$= 0 \text{ , otherwise} \qquad (6.14)$$

From equation (6.11) α cut of $\hat{a}$ is

$\hat{a}^{(\alpha)}$ (=)[$0.00049\alpha + 0.36232, \text{-}0.00054\alpha + 0.36335$]

Thus f.m.f is

$$\mu_{\hat{a}}(x) = \frac{x+0.36232}{0.00049} \quad , 0.36232 \le x \le 0.36281$$
$$= \frac{-x+0.36335}{0.00054} \quad , 0.36281 \le x \le 0.36335$$
$$= 0, \text{ otherwise} \qquad (6.15)$$

From equation (6.12) α cut of $\hat{A}$ is

$A^{(\alpha)}$ (=)[$\exp(0.00002\alpha + 6.085), \exp(-0.00003\alpha + 6.08505)$]

Thus f.m.f is

$$\mu_{\hat{A}}(x) = \frac{\ln x - 6.085}{0.00002} \quad , 439.219 \le x \le 439.229$$
$$= \frac{-\ln x + 6.8505}{0.00003} \quad , 439.229 \le x \le 439.242$$
$$= 0 \text{ , otherwise} \qquad (6.1.6)$$

From the equation (6.13) the α cut and f.m.f of estimated population for 1961 to 2001 are given below -

For the year 1961 i.e. $t = 0$ the α cut is

$P(0)^{(\alpha)}$ (=)[$\exp(0.00002\alpha + 6.085), \exp(\text{-}0.00003\alpha + 6.08505)$]

Thus f.m.f is

$$\mu_{P(0)}(x) = \frac{\ln x - 6.085}{0.00002} \quad , 439.219 \le x \le 439.229$$
$$= \frac{-\ln x + 6.08505}{0.00056} \quad , 439.229 \le x \le 439.242$$
$$= 0, \text{ otherwise} \qquad (6.1.7)$$

For the year 1971 i.e. $t = 0.5$ the α cut is

$P(0.5)^{(\alpha)}$ (=)[$\exp(0.00057\alpha + 6.31215), \exp(\text{-}0.00056\alpha + 6.31328)$]

Thus f.m.f is

$$\mu_{P(0.5)}(x) = \frac{\ln x - 6.31215}{0.00057}, \quad 551.2288 \le x \le 551.549$$
$$= \frac{-\ln x + 6.31328}{0.00056}, \quad 551.549 \le x \le 551.8521$$
$$= 0, \text{ otherwise} \qquad (6.1.8)$$

For the year 1981 i.e. $t = 1.0$ the α cut is

$P(1.0)^{(\alpha)} (=) [\exp(0.00102\alpha + 6.52596), \exp(-0.001\alpha + 6.52798)]$

Thus f.m.f is

$$\mu_{P(1.0)}(x) = \frac{\ln x - 6.52596}{0.00102}, \quad 682.635 \le x \le 683.331$$
$$= \frac{-\ln x + 6.52798}{0.001}, \quad 683.331 \le x \le 684.015$$
$$= 0, \text{ otherwise} \qquad (6.1.9)$$

For the year 1991 i.e. $t = 1.5$ the α cut is

$P(1.5)^{(\alpha)} (=) [\exp(0.00143\alpha + 6.73243), \exp(-0.00142\alpha + 6.73528)]$

Thus f.m.f is

$$\mu_{P(1.5)}(x) = \frac{\ln x - 6.73243}{0.00143}, \quad 839.184 \le x \le 840.385$$
$$= \frac{-\ln x + 6.73528}{0.00142}, \quad 840.385 \le x \le 841.579$$
$$= 0, \text{ otherwise} \qquad (6.1.10)$$

For the year 2001 i.e. $t = 2.0$ the α cut is

$P(2.0)^{(\alpha)} (=) [\exp(0.0018\alpha + 6.93428), \exp(-0.0018\alpha + 6.93788)]$

Thus f.m.f is

$$\mu_{P(2.0)}(x) = \frac{\ln x - 6.93428}{0.0018}, \quad 1026.879 \le x \le 1028.729$$
$$= \frac{-\ln x + 6.93788}{0.0018}, \quad 1028.729 \le x \le 1030.583$$
$$= 0, \text{ otherwise} \qquad (6.1.11)$$

Taking $t = 2$ the equation (6.14) gives the α cut and f.m.f of projected population for 2011, 2016 and 2021 as follows.

The α cut of projected population for the year 2011, where $\eta = 0.5$ is

$P(2.5)^{(\alpha)} (=) [\exp(0.00037\alpha + 7.134726), \exp(-0.000377\alpha + 7.135473)]$

Thus f.m.f is

$$\mu_{P(2.5)}(x) = \frac{\ln x - 7.134716}{0.00037}, \quad 1254.781 \le x \le 1255.257$$
$$= \frac{-\ln x + 7.135473}{0.000377}, \quad 1255.257 \le x \le 1255.731$$
$$= 0, \text{ otherwise} \qquad (6.1.12)$$

The α cut of projected population for the year 2016 is

$P(2.75)^{(\alpha)} (=) [\exp(0.00054\alpha + 7.23314), \exp(-0.00056\alpha + 7.23424)]$

Thus f.m.f is

$$\mu_{P(2.75)}(x) = \frac{\ln x - 7.23314}{0.00054} \quad , \ 1384.563 \le x \le 1385.311$$
$$= \frac{-\ln x + 7.23424}{0.00056} \quad , \ 1385.311 \le x \le 1386.087$$
$$= 0, \text{ otherwise} \qquad (6.1.13)$$

The α cut of projected population for the year 2021 is

$P(3)^{(\alpha)}$ (=)[$\exp(0.00071\alpha + 7.33104), \exp(-0.00073\alpha + 7.33248)$]

Thus f.m.f is

$$\mu_{P(3)}(x) = \frac{\ln x - 7.33104}{0.00071} \quad , \ 1526.969 \le x \le 1528.054$$
$$= \frac{-\ln x + 7.33248}{0.00073} \quad , \ 1528.054 \le x \le 1529.169$$
$$= 0, \text{ otherwise} \qquad (6.1.14)$$

The fuzzified estimated populations of India since 1961 to 2001 are

Table.1.5
(Fuzzified estimated population)

year	$P(\hat{t})$
1961	(439.219,349.229,439.242)
1971	(551.229,551.549,551.852)
1981	(682.635,683.331,684.015)
1991	(839.184,840.385,841.579)
2001	(1026.879,1028.729,1030.583)

The fuzzified projected populations of India for the year 2011, 2016 and 2021 are

Table. 1.6
(Fuzzified projected population)

year	$P(t+\eta)$
2011	(1254.781,1255.257,1255.731)
2016	(1384.563,1385.311,1386.087)
2021	(1526.969,1528.054,1529.169)

7. CONCLUSION

From the evaluation we would say that we fuzzified the model with hyperbolic relative growth rate. The computation by the first method is simple

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Comparison of approximations for area under the standard normal curve

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Abstract: There are numerous approximations to standard normal cumulative distribution function. In this paper, some approximations are compared and the best among them is identified.

Keywords: Cumulative distribution function, approximation, normal distribution.

Introduction

The normal distribution plays a central role in statistics. Closed form representation does not exist for cumulative distribution function (cdf) of normal distribution. This inconvenience in readily evaluating the cdf led to a few attempts at constructing approximations. Moreover academicians and practitioners are often required to write programs in different languages (FORTRAN, C etc.) which require the cdf of normal distribution. Presently, libraries of such programming languages do not offer any in built subroutine or function to compute the cdf. So, an approximate formula can be of use in such situation.

Normal cdf approximations can be broadly classified into three types as given below

(i) Construction of approximate formulas

(ii) Use of approximation of other distribution to normal distribution

(iii) Construction of bounds

The first category of approximation "Construction of approximate formulas" is the most popular. It can further be classified as follows:

(a) a single approximation formula for the entire range of the standard normal variate x.

(b) two or more than two approximation formulas for different ranges of the standard normal variate x.

The subject of approximating areas under the standard normal curve is more than a century old. Several approximations to the normal distributions exist in literature, but most of these approximations are hard to keep in mind and are not simple enough to be computationally attractive. In this article, comparisons are made for single approximation formulas of the standard normal cdf.

Literature Review

Single approximation formulas for standard normal cdf are placed below:

1. Polya(1949): $\Phi(x) \approx \frac{1}{2}\left[1+\left\{1-\exp\left(-\frac{2x^2}{\pi}\right)\right\}^{\frac{1}{2}}\right]$

2. Cadwell (1951)-A :

$$\Phi(x) \approx \frac{1}{2}\left[1+\left\{1-\exp\left(-\frac{2x^2}{\pi}+\frac{2}{3\pi^2}(\pi-3)x^4\right)\right\}^{\frac{1}{2}}\right]$$

3. Cadwell (1951)-B

$$\Phi(x) \approx \frac{1}{2}\left[1+\left\{1-\exp\left(-\frac{2x^2}{\pi}+\frac{2}{3\pi^2}(\pi-3)x^4-0.0005x^6+0.00002x^8\right)\right\}^{\frac{1}{2}}\right]$$

4. Tocher (1963): $\Phi(x) \approx e^{2kx}/(1+e^{2kx})$ where $k = \sqrt{2/\pi}$

5. Zelen and Severo (1964):

$$\Phi(x) \approx 1-(0.4361836\,t-0.1201676\,t^2+0.9372980\,t^3)\left(\sqrt{2/\pi}\right)^{-1} e^{-x^2/2},$$

where $t = (1+0.33267x)^{-1}$

6. Page (1977): $\Phi(x) \approx 0.5\{1+\tanh(y)\}$ where $y = \sqrt{2/\pi}\,x(1+0.044715x^2)$

7. Hammakar (1978): $1-\Phi(x) \approx 0.5\left\{1-\left(1-e^{-y^2}\right)^{0.5}\right\}$, $y = 0.806x(1-0.018x)$

8. Lin (1989): $1-\Phi(x) \approx 0.5\left(e^{-0.717x-0.416x^2}\right)$, $x > 0$

9. Lin (1990): $1-\Phi(x) \approx 1/(1+e^y)$ where $y = 4.2\,\pi\{x/(9-x)\}$, $x > 0$

10. Bagby (1995):

$$\Phi(x) \approx 0.5+0.5\left[1-\left(\frac{1}{30}\right)\left\{7e^{-x^2/2}+16e^{-x^2(2-\sqrt{2})}+\left(7+\frac{\pi}{4}x^2\right)e^{-x^2}\right\}\right]^{0.5}, x>0$$

11. Shore (2005): $\Phi(x) \approx \frac{1+g(-x)-g(x)}{2}$

where $g(x) = \exp[-\log(2)\exp[\{\alpha/(\lambda/S_1)\}\{(1+S_1x)^{(\lambda/S_1)}-1\}+S_2x]]$

$\lambda = -0.61228883, S_1 = -0.11105481, S_2 = 0.44334159, \alpha = -6.37309208$

12. Aludaat and Alodat (2008): $\Phi(x) \approx 0.5 + 0.5\sqrt{1-e^{-\sqrt{\frac{\pi}{8}}x^2}}$

13. Winitzki (2008): $\Phi(x) \approx \dfrac{1+\left[1-\exp\left(-\dfrac{x^2}{2}\dfrac{\frac{4}{\pi}+0.147\frac{x^2}{2}}{1+.147\frac{x^2}{2}}\right)\right]^{\frac{1}{2}}}{2}$

To determine the accuracy of the above formulas, maximum absolute errors have been computed for each of the approximations for x =0.0005 (0.0005) 4. One can take cdf as 1 for $x > 4$.

Out of the three approximations proposed by Shore (2005), computation has been performed for the third one because it is the most accurate by his own admission. Winitzki (2008) has proposed two approximations for error function out of which the first approximation is less accurate than the second one. The improved error function approximation given by Winitzki (2008) is

$$erfx \approx \left[1-\exp\left(-x^2\frac{\frac{4}{\pi}+ax^2}{1+ax^2}\right)\right]^{\frac{1}{2}}, \text{ where } a = 0.147.$$

Using the relation between error function and normal cdf, Winitzki (2008) approximation to normal cdf given in (13) above can be obtained.

Comparison of approximations

The following table shows maximum absolute error calculated for different approximate formulas.

Approximation Formula	Maximum Absolute Error for Range of the Standard Normal Variable (0, 4]
Polya (1949)	3.1382E-03
Cadwell (1951)-A	6.6714E-04
Cadwell (1951)-B	4.7628E-05
Tocher (1963)	1.7671E-02
Zelen and Severo (1964)	1.1176E-05
Page (1977)	1.7913E-04
Hammakar (1978)	6.2294E-04
Lin (1989)	6.5853E-03
Lin (1990)	6.6876E-03
Bagby (1995)	3.0372E-05
Shore (2005)	6.6074E-07
Aludaat and Alodat (2008)	1.9732E-03
Winitzki (2008)	6.2030E-05

From the table we find that Shore (2005) formula is the best formula in terms of maximum absolute error.

Conclusion

The desirable properties of standard normal cdf approximations are

1. Simplicity
2. Have very good accuracy
3. Be invertible

Aludaat and Alodaat (2008) formula is easily invertible, but is not very accurate. Shore (2005) approximation formula provides minimum accuracy of six decimal digits, but is not easily invertible. In case if our emphasis lies on accuracy, Shore (2005) formula is to be used.

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